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Symplectic Interpretation for the Discretization of Some Physical Magnitudes

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We study some conditions for a pullback of a symplectic nonvectorial fiber bundle to have a discrete holonomy group. By applying this structure to the relativistic massless free particles for a hypersurface of simultaneity, and by using symplectic mechanics methods, we obtain that some physical magnitudes take discrete values.

1. INTRODUCTION

The Aharonov–Bohm experiment was of great significance in quantum physics (Wu and Yang, 1975). It showed that the phase of the electron wave function, when a beam of monoenergetic electrons is diffracted around a solenoid, is quantized according to Dirac's theory.

A geometrical explanation of this phenomenon can be obtained by means of the fiber bundle theory. Actually, a principal fiber bundle is used with structural group $U(1)$ that permits us to interpret the phase discretization by the fact that the closed trajectories generate a discrete holonomy group (Moriyasu, 1983).

Wu and Yang (1975) and Moriyasu (1983) described completely the geometrical meaning of the above experiment. However, if we want to obtain a geometrical interpretation for the discretization of some physical magnitudes for massless elementary free particles, we need to overcome two difficulties:

(a) A principal bundle characterizes the interaction, but it does not provide any information about the particles involved.

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(b) The Aharonov–Bohm experiment deals with electron beams, i.e., at every instant the observer sees an electron beam that bifurcates before going around an electromagnet and afterward these branches impact on the same point.

In order to solve problems of the first type, we constructed nonvectorial associated fiber bundles provided with a connection, which we called *seeded fiber bundles* (Liern and Olivert, 1995a, b). In such bundles the motion structure of the particles, a symplectic structure (Souriau, 1970a), is combined with a connection that expresses the gravitational interaction given by general relativity.

On the other hand, if the seeded fiber bundle base is a space–time manifold $\mathcal{M}$ (Sachs and Wu, 1983), the simultaneity condition mentioned in (b) cannot be achieved. Therefore, it is necessary to design methods to work on a simultaneity hypersurface, Landau's manifold (Olivert, 1980), in such a way that most of the properties in $\mathcal{M}$ are preserved. So we use the idea of a pullback bundle for the seeded fiber bundles. We show that some physical magnitudes are transformed in such a way that they only admit a discrete number of values.

2. A NEW SYMPLECTIC FORMALISM: *SEEDED FIBER BUNDLES*

Usually, the evolution of a dynamical system is studied over a space–time manifold. However, in this paper we want to treat both the internal symmetries and the interactions. Therefore we need to consider a principal G bundle $\lambda = (P, B, \pi, G)$ with a connection $\mathcal{H}$. The group G represents the local symmetries and the connection represents the interaction. The dynamical system to consider is (F, σ_F), a symplectic Hausdorff manifold left G space. If we are interested in the behavior of F in λ, we can construct $\lambda[F] = (P_F, B, \pi_F, G)$, a fiber bundle associated to λ with fiber type F. In order to obtain a more useful structure we need to set some requirements.

Definition 1. The fiber bundle $\lambda[F] = (P_F, B, \pi_F, G)$ described above is a seeded fiber bundle (SFB) if there exists a foliation $\mathcal{S}$ contained in the horizontal distribution such that:

(i) P_F admits an atlas with flat charts (Brickell and Clark, 1970) with respect to S and $\mathcal{R} := \ker \pi_{F*p}$.

(ii) Every fiber intersects every leaf at most at one point.

These bundles will be denoted by $\lambda[F](S) = (P_F, B, \pi_F, G; S)$.

Under these conditions, the foliation S can be projected to a foliation Ω over B:

$$\Omega := \{X | X \text{ is a vector field in } B\text{: } \pi_{F*}Y = X \circ \pi_F,\ Y \in S\} \tag{1}$$

such that dim Ω = dim S. If B has a connection, the integral manifolds of Ω will allow us to describe the motion over B. Let X be a vector field of Ω, c a maximal integral curve of X, and τ_t^c: $T_{c(0)}\ B \to T_{c(m)}\ B$ the parallel displacement along $c(t)$ $\forall t \in I$, where I is the domain of c. We say that $\lambda[F]$ is provided with a *motion law* if

$$\tau_t^c \Omega\ (c(0)) = \Omega(c(t)) \qquad (t \in I) \tag{2}$$

(a) If Ω is a 1-foliation, the motion law gives rise to geodesics in B (Liern and Olivert, 1995b).

(b) If Ω is a 3-foliation, the motion law gives rise to totally geodesic integral manifolds in B (Liern and Olivert, 1995b).

This definition can be characterized in a more useful way as follows (Liern and Olivert, 1995a).

Theorem 1. Let $\lambda = (P, B, \pi, G)$ be a principal G bundle with a connection $\mathcal{H}$ and let (F, σ_F) be a symplectic Hausdorff manifold left G space. A fiber bundle $\lambda[F] = (P_F, B, \pi_F, G)$ associated to λ with fiber type F is SFB if and only if for every $m \in B$ there exists a presymplectic regular manifold $(V_m, \sigma_m) \subset P_F$ satisfying:

(a) dim $V_m = k$ (constant) $\forall m \in B$.

(b) There is a surjective submersion ψ_m: $V_m \to \pi_F^{-1}\ (m)$ such that $\pi_F^{-1}\ (m) = V_m/\ker \psi_{m*}$.

(c) Given m, $n \in B$, if $V_m \cap V_n \neq \emptyset$, then $V_m = V_n$.

(d) $\ker(\sigma_m)_w \subset \mathfrak{Q}_w$, where $\mathfrak{Q}$ is the horizontal distribution in P_F induced by $\mathcal{H}$, and $w \in V_m$.

Example. For $i = 1, 2$, let $\lambda_i[U_i] = (P_{U_i}, B_i, \pi_i, G_i)$ be a fiber bundle associated to a principal G_i bundle λ_i with fiber type (U_i, σ_i), a symplectic manifold left G_i space. The product

$$\xi[U_1 \times U_2]: = (P_{U_1} \times P_{U_2}, B_1 \times B_2, \pi_1, \times \pi_2, G_1 \times G_2)$$

is a SFB. So, let us consider the Ehresmann connection λ_i $(i = 1, 2)$ in λ_i, and $\mathcal{H}_1 \times \mathcal{H}_2$ in $\xi = \lambda_1 \times \lambda_2$. Given $(m_1, m_2) \in B_1 \times B_2$, we construct the manifold

$$V_{(m_1,m_2)} = P_{U_1} \times \pi_2^{-1}(m_2)$$

The symplectic 2-form of the fiber $\pi_1^{-1}(m_1) \times \pi_2^{-1}(m_2) \subset V_{(m_1,m_2)}$ induces a presymplectic 2-form in $V_{(m_1,m_2)}$. The family $\{(V_{(m_1,m_2)}, \sigma_{(m_1,m_2)})\}_{(m_1,m_2)\in B_1\times B_2}$ satisfies Theorem 1.

Remarks. (a) If we consider the family

$$V_{m_1 \times m_2} := \pi_1^{-1}(m_1) \times P_{U_2}, \qquad (m_1, m_2) \in B_1 \times B_2, \quad \text{in} \quad \xi[U_1 \times U_2]$$

Theorem 1 also holds, showing the nonunicity of the "seeded structure."

(b) If λ_1 or λ_2 is a nontrivial bundle, then $\xi[U_1 \times U_2]$ is a nontrivial bundle.

This example provides a method to construct an SFB. However, from a physical point of view we are interested in the evolution of dynamical systems over one space–time manifold (not over a product) or even more generally over manifolds embedded in it. Hence, we need to see what happens if we "reduce" the base manifold of an SFB.

Given a differentiable manifold L and an injective immersion f: $L \to B$, we consider $f^*(\lambda[F](S)$, the pullback bundle of $\lambda[F](S)$. By construction the diagram

$$\begin{array}{ccc} f^*(P_F) & \xrightarrow{pr_2} & P_F \\ {\scriptstyle pr_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_F} \\ L & \xrightarrow{f} & B \end{array}$$

is commutative, where

$$\begin{aligned} f^*(P_F) &= L \times_B P_F = \{(m, x) \in L \times P_F: f(m) = \pi_F(x)\} \\ &= \bigcup_{m \in L} (\{m\} \times \pi_F^{-1}((f(m))) \end{aligned}$$

$$pr_1(m, x) = m, \qquad pr_2(m, x) = x, \qquad \forall (m, x) \in f^*(P_F) \tag{3}$$

It is easy to prove the following properties:

Lemma 1. (a) pr_2 is injective.

(b) If v is a vertical vector in $f^*(P_F)$, then $pr_{2*}(v)$ is a vertical vector in P_F.

Let $\mathcal{L} := f(L) \subset B$ be an immersed submanifold of B. We define the correspondence Ω_1: $\mathcal{L} \to \mathcal{P}(T\mathcal{L})$ as

$$\Omega_1(f(m)) = \Omega(f(m)) \cap T_{f(m)}\mathcal{L}, \qquad \forall m \in L \tag{4}$$

where Ω is the p-foliation on B given by (1).

Lemma 2. If $\dim \Omega_1(f(m)) = k_o$ (constant) $\forall m \in L$, then

$$\Omega^* := \{f_*^{-1}(X_1): X_1 \in \Omega_1\} \tag{5}$$

is a k_o-foliation on L.

Proof. It can be checked that Ω_1 is a k_o-distribution on $\mathcal{L}$. Next, we prove that Ω_1 is involutive. Let X_1, Y_1 be two vector fields in Ω_1; then there are two vector fields $\tilde{X}$, $\tilde{Y}$ such that

$$\pi_{F*}\tilde{X} = X_1 \circ \pi_F, \qquad \pi_{F*}\tilde{Y} = Y_1 \circ \pi_F$$

Then,

$$[X_1, Y_1] \circ \pi_F = \pi_{F*}[\tilde{X}, \tilde{Y}] \subset \pi_{F*}S = \Omega \tag{6}$$

As X_1, X_2 are tangent to $\mathcal{L}$, by construction $[X_1, X_2]$ is tangent to $\mathcal{L}$ and for (6), $[X_1, X_2] \subset \Omega_1$. Moreover, as Ω_1 is a k_o-foliation of $\mathcal{L}$ and f: $L \rightarrow \mathcal{L}$ is a diffeomorphism, then Ω^* is a k_o-foliation on L. ■

By Lemmas 1 and 2 and (4) we have the following result:

Theorem 2. Let $\lambda[F](S) = (P_F, M, \pi_F, G; S)$ be an SFB, L a differentiable manifold, and f: $L \rightarrow B$ an injective immersion. If $\dim \Omega_1(f(m)) = k_o$(constant) $\forall m \in L$, then $f^*(\lambda[F](S))$ is an SFB.

Proof. Let us consider

$$S^* := \{X' = (f_*^{-1}X_1, \tilde{X}),\ X_1 \in \Omega_1,\ \tilde{X} \in S \text{ such that } \pi_{F*}\tilde{X} = X_1\} \tag{7}$$

which is nonempty by Lemma 2. We will proove that S^* is a k_o-foliation on $f^*(P_F)$. Let X', $Y' \in S^*$; then a little algebraic manipulation leads to

$$[X', Y'] = [(f_*^{-1}X_1, \tilde{X}), (f_*^{-1}Y_1, \tilde{Y})] = (f_*^{-1}\,[X_1, Y_1], [\tilde{X}, \tilde{Y}])$$

Notice that $[f_*^{-1}X_1, \tilde{Y}] = [\tilde{X}, f_*^{-1}Y_1] = 0$, because every term in the Lie bracket depends on different variables.

Let $X' \in S^*$; we have that

$$pr_{2*}X' = pr_{2*}(f_*^{-1}X_1, \tilde{X}) = \tilde{X} \circ pr_2 \in \mathcal{S} \tag{8}$$

X' can be split as $X' = X'_H + X'_V$ where X'_H (resp. X'_V) denotes the horizontal (resp. vertical) part of X'. If $X'_V \neq 0$, by Lemma 1, $pr_{2*}X'$ would have a vertical part. But by Definition 1, $\mathcal{S}$ is horizontal, and by (8), $pr_{2*}X' \in \mathcal{S}$, and we would get a contradiction. Therefore X' is horizontal.

Now, let H^* be a leaf of S^*; by (8) and Lemma 1, there exists a leaf H of $\mathcal{S}$ such that

$$pr_2(H^* \cap pr_1^{-1}(m)) \subset H \cap \pi_F^{-1}(f(m)) \tag{9}$$

As $\lambda[F](S)$ is an SFB, then $H^* \cap pr_1^{-1}(m)$ is empty or a singleton.

Further, if $\{h_i\}_{i\in I}$ is an atlas of P_F with flat charts for the foliation $\mathcal{S} \times \mathcal{R}$, then $\{f^{-1}h_{i|\mathcal{L}}\}_{i\in I}$ is an atlas of $f^*(P_F)$ with flat charts for the foliation $S^* \times \mathcal{R}$. ■

Under the assumptions in Theorem 2, $\lambda[F](S)$ and $f^*(\lambda[F])(S^*)$ are SFBs. Theorem 1 guarantees the existence of presymplectic manifolds $\{(V_m, \sigma_m)\}_{m \in B}$ and $\{(\tilde{V}_n, \tilde{\sigma}_n)\}_{n \in L}$ contained in P_F and $f^*(P_F)$, respectively. The manifolds

$$E = \bigcup_{m \in B} V_m, \qquad E^* = \bigcup_{n \in L} \tilde{V}_n \tag{10}$$

are also presymplectic manifolds (Liern and Olivert, 1995b) with Lagrange forms σ and $\tilde{\sigma}$, respectively, satisfying

$$\sigma_{|V_m} = \sigma_m, \quad \forall m \in B, \qquad \tilde{\sigma}_{|\tilde{V}_n} = \tilde{\sigma}_n, \quad \forall n \in L \tag{11}$$

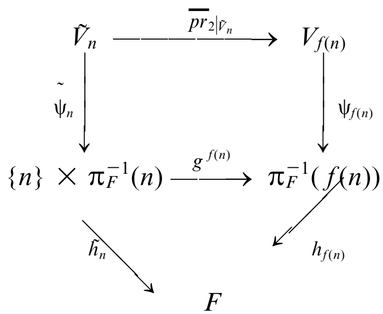
Proposition 1. Let (E, σ) and $(E^*, \tilde{\sigma})$ be the presymplectic manifolds defined by (10) and (11). Then $\overline{pr}_2^* \sigma = \tilde{\sigma}$, where $\overline{pr}_2$: $E^* \to E$ is the map induced by pr_2.

Proof. Let us consider the SFBs $\lambda[F](S)$ and $i^*(\lambda[F])(S^*)$, where i is the canonical immersion. By Theorem 1 we have that

$$S_{|V_m} = \ker \sigma_{|V_m} = \ker \sigma_m, \qquad m \in B \tag{12}$$

$$S^*_{|\tilde{V}_n} = \ker \tilde{\sigma}_{|\tilde{V}_n} = \ker \tilde{\sigma}_n, \qquad n \in L$$

For $n \in L$ the diagram



is commutative, where $g_n^{f(n)}$, $\tilde{h}_n$, $h_{f(n)}$ are diffeomorphisms, and $\tilde{\psi}_n$ and $\psi_{f(n)}$ are surjective submersions satisfying part (b) of Theorem 1.

The maps

$$\tilde{\varphi}_n: = \tilde{h}_n \circ \tilde{\psi}_n, \qquad \varphi_{f(n)} := \tilde{h}_{f(n)} \circ \psi_{f(n)} \tag{13}$$

are also surjective submersions such that

$$F = \frac{\tilde{V}_n}{\text{Ker } \tilde{\varphi}_{n*}}, \qquad F = \frac{V_{f(n)}}{\text{Ker } \varphi_{f(n)*}}$$

By (8) and (12), $\varphi_{f(n)} \circ \overline{pr}_2 = \tilde{\varphi}_n$. As $\sigma_n = \varphi_n^* \sigma_F$ and $\sigma_{f(n)} = \varphi_{f(n)}^* \sigma_F$, we have that $\overline{pr}_2^* \sigma_{|v_{f(n)}} = \tilde{\sigma}_{|v_n}$. As this can be done for all $n \in L$, and as E and E^* are disjoint unions of manifolds, we have that $\overline{pr}_2^* \sigma = \tilde{\sigma}$. ■

3. COUNTABLE HOLONOMY GROUPS AND SEEDED FIBER BUNDLES

The above results show when the seeded structure is maintained via pullback. However, if our purpose is to extend the idea of Wu and Yang (1975) in the sense that a principal G bundle is "responsible" for the discretization, we must ask some extra requirements for the principal G bundle that gives rise to the SFB.

Given $\lambda = (P, B, \pi, G)$ a principal G bundle, let L be a differentiable manifold and f: $L \to B$ an injective immersion; we can consider $f^*(\lambda) = (f^*(P), L, p_1, G)$ the pullback of λ, where

$$f^*(P) = L \times_B P, \qquad p_1(l, p) = l, \qquad (l, p) \in L \times_B P \tag{14}$$

The map p_2: $f^*(P) \to P_F$, given by

$$p_2(l, p) = p, \qquad (l, p) \in L \times_B P \tag{15}$$

is differentiable and satisfies $\pi \circ p_2 = f \circ p_1$ (Kobayashi and Nomizu, 1963).

In the principal G bundle $f^*(\lambda)$ there is a unique connection $\mathcal{H}^*$ such that $p_{2*}(\mathcal{H}^*) \subset \mathcal{H}$ (Kobayashi and Nomizu, 1963). In addition, if $w_{\mathcal{H}}$, $\Omega_{\mathcal{H}}$ are the connection form and the curvature form of $\mathcal{H}$, respectively, then

$$p_2^* \, \Omega_{\mathcal{H}} = \Omega_{\mathcal{H}^*}, \qquad p_2^* \, w_{\mathcal{H}} = w_{\mathcal{H}^*} \tag{16}$$

where $\Omega_{\mathcal{H}^*}$, $w_{\mathcal{H}^*}$ are the curvature form and the connection form of $\mathcal{H}^*$, respectively.

Theorem 3. Let $\lambda = (P, B, \pi, G)$ be a principal G bundle with a connection $\mathcal{H}$, and let $f^*(\lambda)$ be the pullback of λ given by (15). If

$$p_{2*}(\mathcal{H}^*_{(m,p)}) \subset \text{Ker } d(w_{\mathcal{H}})_p, \qquad \forall (m, p) \in f^*(P) \tag{17}$$

then the holonomy group of $f^*(\lambda)$ is countable.

Proof. Let ϕ and ϕ^0 be the holonomy group and the restricted holonomy group of $\mathcal{H}^*$, respectively. Since L is a connected manifold and admits a countable basis, ϕ/ϕ^0 is countable.

On the other hand, let X', Y' be two vector fields of $f^* \, P$; we have that

$$\begin{aligned}\Omega_{\mathcal{H}^*}(X', Y') &= dw_{\mathcal{H}^*}(p_{\mathcal{H}^*}X', p_{\mathcal{H}^*}Y') = dp_2^* w_{\mathcal{H}}(p_{\mathcal{H}^*}X', p_{\mathcal{H}^*}Y') \\ &= p_2^* dw_{\mathcal{H}}(p_{\mathcal{H}^*}X', p_{\mathcal{H}^*}Y') = dw_{\mathcal{H}}(p_{2*}p_{\mathcal{H}^*}X', p_{2*}p_{\mathcal{H}^*}Y') = 0\end{aligned}$$

By the Ambrose–Singer theorem (Choquet-Bruhat *et al.*, 1978), the Lie algebra of ϕ is trivial. Then, so we have that $\phi^0 = \{e\}$; therefore $\phi/\{e\} = \phi$ is countable. ■

Remark. Every trivial principal G bundle with flat connection obviously satisfies (17).

Husemoller (1966) proved that the fiber bundle $f^*(\lambda)[F] = (f^*(P)_F, L, p_{1_F}, G)$ associated to $f^*(\lambda)$ with fiber type (F, σ_F) is L-isomorphic to the SFB $f^*(\lambda[F](S))$ described in (3). Therefore, under the assumptions in Theorem 2, $(f^*\lambda)[F]$ is an SFB. Then, we have an SFB generated by a principal G bundle with a countable holonomy group. Now we can apply this to elementary free particles.

4. SIMULTANEITY AND MASSLESS ELEMENTARY FREE PARTICLES

Souriau (1970a, b) introduced the concept of massless elementary free particles as a symplectic manifold (U, σ_U) of dimension 6 which admits Poincaré's restricted group G_o, as dynamic and transitive group. Moreover, (U, σ_U) is diffeomorphic to $\mathrm{IR}^4 \times S^2$ and symplectomorphic to an orbit of the coadjoint representation of G_o. in its coalgebra, $T_e^* G_o$.

Let M_4 be the space–time of Minkowski, and η the Minkowski tensor. In (M_4, η), a massless elementary free particle is characterized by

$$\eta(\mathbf{p}, \mathbf{p}) = \eta(\mathbf{w}, \mathbf{w}) = 0 \tag{18}$$

where $\mathbf{p}$ is the energy-momentum vector and $\mathbf{w}$ is the polarization of the particle.

Let χ, s be the particle's helicity and its spin, respectively; then $\mathbf{w} = \chi s \mathbf{p}$. There exists an isotropic vector Y satisfying $*M_x = \mathbf{p} \wedge Y$, where $*M_x$ is the dual of the Lorentz momentum. The evolution space of (U, σ),

$$V := \{y = (x, I, J), I = p, J = \chi Y/s\} \tag{19}$$

is a Hausdorff nine-dimensional manifold (Souriau, 1970a; Grigore and Popp, 1992). Furthermore, there is a surjective submersion θ: $V \to U$ given by $\theta(y) = (si_I iJ\nu + x \wedge I, I)$, where ν is the element of volume of (M_4, η). The manifold V can be provided with a presymplectic form $\sigma_V = \theta^*\sigma_U$, and its characteristic foliation $S := \mathrm{Ker}(\sigma_V)_b$ has dimension 3.

Liern and Olivert (1995b) proved the following:

Theorem 4. Let $\tilde{\xi} = (E, M_4, \tilde{\pi}, G_o)$ be a principal G_o, bundle over M_4 (the Minkowski space–time) with structural group G_o, with the fiber type (U, σ) given above. Then:

(a) $\tilde{\xi}[U](S) = (E_U, M_4, \tilde{\pi}_U, G_o; S)$ is an SFB that we call the Souriau fiber bundle.

(b) The motion law given by (2) gives rise to isotropic hyperplanes in M_4.

If we extend $\tilde{\xi}[U](S) = (E_U, M_4, \tilde{\pi}_U, G_o; S)$ to an SFB

$$\xi[U](S) = (P_U, \mathcal{M}, \pi_U, G_o; S \tag{20}$$

with base manifold $\mathcal{M}$ (the space–time manifold), such that $\xi[U](S)$ is locally isomorphic to $\tilde{\xi}[U](S)$, the motion law given by (2) gives rise to totally geodesic integral manifolds of dimension 3 in $\mathcal{M}$

4.1. Pullback Bundles and Massless Free Elementary Particles

Let $C \in \mathcal{M}$ be an observer, p a position in C, and u_p the 4-velocity of C in p. It has been proved (Olivert, 1980) that there is a unique regular three-dimesional submanifold of $\mathcal{M}$ L_p (called a *Landau manifold*) such that the physical space M_p is tangent to L_p in p, and whose points are simultaneous to p in standard time.

Let (U, σ_U) be a massless elementary free particle and L_p a Landau manifold. According to (3), we can construct the pullback of $\xi[U](S) = (P_U, \mathcal{M}, \pi_U, G_o; S)$, determined by the canonical inclusion i: $L_p \to \mathcal{M}$, which is denoted by $i^*(\xi[U]) = (P^*_U, L_p, (pr_1)_U, G_o)$.

We have shown (Liern and Olivert, 1995b) that the map $\mathcal{G}$: $L_p \to \mathcal{P}(TL_p)$ given by

$$\mathcal{G}(m) := \Omega(m) \cap i_*(T_m L_p), \qquad m \in L_p \tag{21}$$

is a 2-foliation on L_p.

By (20) and Theorem 2, $i^*(\xi[U])$ is an SFB that can be rewritten as $i^*(\xi[U])(S^*) = (P^*_U, L_p, (pr_1)_U, G_o; S^*)$.

On the other hand, we can consider the pullback of the principal bundle ξ, in the same way as in (14) $i^*(\xi) = (P^*, L_p, p_1, G_o)$, where

$$P^* = L_p \times_{\mathcal{M}} P = \{(m, x) \in \mathcal{M} \times P: \quad i(m) = \pi(x)\}$$

$$p_1(m, x) = m, \qquad p_2(m, x) = x, \qquad \forall (m, x) \in P^* \tag{22}$$

Let w be the one-form of the connection $\mathcal{H}$ in the principal G bundle ξ and let $\mathcal{H}^*$ be the connection induced in $i^*(\xi)$ (Kobayashi and Nomizu, 1963). We can suppose that $p_{2*}(\mathcal{H}^*_{(m,p)}) \subset \ker dw_p$, $\forall (m, p) \in P^*$, because $\xi[U](S)$ is constructed in such a way that locally satisfies the conditions of the SFB $\tilde{\xi}[U](S)$ given by Theorem 4. As they are trivial bundles, the condition (17) is trivially verified.

As the Landau manifolds admit a countable basis, we can apply Theorem 3; then the holonomy group of $i^*(\xi[U])$ is countable.

4.2. Discretization

The motion law described by (2) does not allow us to work with symplectomorphisms because the space–time manidold is neither presymplectic nor symplectic. In consequence we cannot study the evolution of the wavefronts originating in the massless elementary free particles. However, it is necessary to study physical magnitudes that are preserved in time. Therefore, we want to work in the total space of an SFB because it admits presymplectic structures [see (10)].

In the same way as we define the motion law in $\mathcal{M}$, we can impose a condition (which we call a *stability condition*) by using symplectomorphisms that preserve the symplectic structures.

Let $\xi[U](S) = (P_U, \mathcal{M}, \pi_U, G_o; S)$ be an SFB given by (20). By (1) there exists in $\mathcal{M}$ a 3-foliation π_U-related to S. Let X be a vector field of Ω, c: $I \to \mathcal{M}(I \subset \mathbb{R})$ an integral curve of X, and τ_t^c: $\pi_U^{-1}(c(0)) \to \pi_U^{-1}(c(t))$ the parallel displacement along c in P_U. Given $p \in P_U$, we can define the curve $\beta^c(t, p)$: $I \to P_U$, given by

$$\beta^c(t, p) = \tau_t^c(p), \qquad t \in I, \quad \text{where} \quad \pi_U(p) = c(0) \tag{23}$$

Then for every $p \in \pi_U^{-1}(c(0))$, we have that $\{\beta^c(t, p)\}_{p \in \pi_U^{-1}(c(0))}$ is a family of differentiable curves in P_U.

By construction, $(\partial/\partial t)\, \beta^c(t, p) \subset S\, (\beta^c(t, p))$; then $\{\beta^c(t, p)\}_{p \in \pi_U^{-1}(c(0))}$ induces a family of curves

$$\gamma^c(t, p); \quad I \times E \to E \tag{24}$$

where E is the manifold given by (10).

Definition 2. We say that the SFB $\xi[U](S)$ is *c-stable* if for every $t_0 \in I$, $\gamma^c(t_0, p)$ is a symplectomorphism.

By direct calculation we can prove the following properties:

Proposition 2. Let $\xi[U](S) = (P_U, \mathcal{M}, \pi_U, G_o, S)$ be an SFB.

(a) If in $\xi[U](S)$ there exists a family of presymplectic manifolds $\{(V_x, \sigma_x)\}_{x \in \mathcal{M}}$ satisfying (*a*)–(*d*) of Theorem 1 and these are symplectomorphic, then $\xi[U](S)$ is c-stable for every integral curve of each vector field of Ω.

(b) If $\xi[U](S)$ is c-stable, there exist curves in P_U that preserve the foliation in the presymplectic manifolds in such a way that the fibers coincide with the parallel displacement.

We can prove that the c-stability condition is inheritable via a pullback.

Proposition 3. Let X^* be a vector field of $\mathcal{G}$, c an integral curve of X^*, and $\tilde{c} = i \circ c$. If $\xi[U](S)$ is $\tilde{c}$-stable, then $i^*(\xi[U])(S^*)$ is c-stable.

Proof. We construct

$$\alpha^c(t, (m, x)) := (c(t), \gamma^{\tilde{c}}(t, pr_2\, m, x))) \tag{25}$$

a family of differentiable curves in $P^*_{\tilde{U}}$, where $(t, (m, x)) \in I \times P^*_{\tilde{U}}$ and $\gamma^{\tilde{c}}$ is the family of curves described by (24).

According to (7), $(\partial/\partial t)\ \alpha^c \subset S^*$; therefore we define $\{\rho^c(t, (m, x)\}_{(m,x)\in E^*}$ a family of differentiable curves in E^*.

By Proposition 1, the Lagrange forms of $(E^*, \tilde{\sigma})$ and (E, σ) are related by $\tilde{\sigma} = \overline{pr}^*_2\sigma$, and by construction

$$\overline{pr}_2\rho^c_t = \gamma^{\tilde{c}}_t\overline{pr}_2 \tag{26}$$

Given $t_0 \in I$, as $\xi[U](S)$ is $\tilde{c}$-stable, we have that $(\gamma^{\tilde{c}}_{t0})^*\ \sigma = \sigma$. Then,

$$\begin{aligned}(\rho^c_{t0})^*\ \tilde{\sigma} &= (\rho^c_{t0})^*\ \overline{pr}^*_2\sigma = (\overline{pr}_2\rho^c_{t0})^*\sigma \\ &= (\gamma^{\tilde{c}}_{t0}\ \overline{pr}_2)^*\ \sigma = \overline{pr}^*_2(\gamma^{\tilde{c}}_{t0})^*\ \sigma = \overline{pr}^*_2\sigma = \tilde{\sigma} \quad \blacksquare\end{aligned} \tag{27}$$

Consider X^* a vector field of $\mathcal{G}$ and

$$c\colon\quad [0, T_0] \to L_p,\quad \text{such that}\quad c(0) = c(T_0) = m_0 \in L_p \tag{28}$$

an integral curve of X^* for which $i^*(\xi[U])(S^*)$ is c-stable.

The curve c induces symplectomorphisms in E^* whose restrictions to the fibers coincide with the parallel displacement [see (23)]. Moreover, each curve satisfying (28) defines an element of the holonomy group. Theorem 3 ensures that the holonomy group of $i^*(\xi[U])(S^*)$ is countable. Thus the family of symplectomorphisms has necessarily a countable cardinality.

Thus the set of images of each leaf H^{S^*} of S^* is countable. Applying the Noether theorem to each leaf in this set, we obtain discrete transformed physical magnitudes.

5. DISCUSSION

Conceptually, this paper started from the fact that using a principal $U(1)$ bundle δ (with a connection and a discrete holonomy group), one can describe the quantization of the phase of the electron wave function observed in the Aharonov–Bhom experiment. We showed that this property is satisfied not only by principal bundles. By using a pullback of an SFB, under certain geometrical requirements, we obtain qualitatively that some physical magnitudes take only a discrete number of values. However, whereas in δ the discrete holonomy group corresponds exactly to the Dirac quantization, we cannot assure that the discretization corresponds to a quantization. We do not know any experiment that supports this kind of discretization.

It must be admitted that we have made a geometrical imposition, (17), but it is plausible, as in flat space–time it holds trivially. Given that an SFB aims to preserve most of the properties of a trivial SFB over the Minkowski space–time (Liern and Olivert, 1995b), such an imposition is not very restrictive. Besides, it is reasonable that the conditions to obtain a discretization must not be too general.

Given that (17) conditions the geometrical structure (curvature) of the base manifold of the bundle, in any event nothing has been demanded of the initial fiber bundle, but it has been made in its pullback. In this case the base manifold is not the space–time manifold, but a hypersurface of simultaneity.

Finally, it may be a little surprising that we work with observations on hypersurfaces of simultaneity. However, when it is required that the curves start and finish at the same point, it makes sense to speak about an observation at such a point, understanding that such an observation is instantaneous.

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